MATHEMATICAL AND PHENOMENOLOGICAL CONTINUITY-INDISCERNIBILITY

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In this paper we shall be dealing with the continuity problem from a phenomenological point of view. Mathematical continuum and its review through a phenomenologically stimulated viewpoint constitutes a fundamental, yet largely unexplored terrain of research at the crossroads of phenomenological philosophy and standard or nonstandard mathematical approach. Looking back at the theoretical work of the Prague school of Alternative Set theory and its shift of the horizon approach, the research in nonstandard analysis and its intensional version, IST theory, taking also into account the claims to a mathematical science "imitating" lebenswelt in the Husserlian sense, we have a tendency at least in the last decades to alternative, more "natural" approaches of foundations. Approaches which produce out of their axiomatical structure a novel, nonconventional definition of continuum and the resulting topological properties. In this paper we try to make clear how their structure imitates the shift of the horizon approach in mathematical- phenomenological attitude. Further, we follow phenomenological reduction to an ultimate subjectivity in time consciousness and try to demonstrate the common conceptual traits with the notion of continuum in the above mentioned mathematical theories.

INTRODUCTION

Attempts to provide an alternative conceptual basis to fundamental mathematical ideas like that of countable infinity, uncountability and continuity are known to have come almost shortly after the Cantor-Dedekind construction of infinite cardinalities and the real number system. Mathematicians of the stature of Kurt Gödel and Hermann Weyl are referred to as being influenced by Husserlian thought – regarding in particular the notion of the continuum – by Dagflinn Føllesdal (Føllesdal, 1999) and Giuseppe Longo (Longo, 2001).[1]

[1] In his review of S. Feferman's paper, "Weyl vindicated: Das Kontinuum 70 years later" Longo 1993, G. Longo stresses that: "what really interests Weyl is the understanding of mathematics as part of our human endeavour towards knowledge, in particular of the physical world. Weyl stresses the inadequacies of the mathematical formalization with respect to a crucial aspect of our physical experience: our intuition of the continuity of space and time (Weyl, in fact, contributed greately to the mathematics of general relativity). In his view, the phenomenal experience of time, as past, present and future, is unrelated to the mathematical treatment of the real numbers. Time cannot be decomposed in points. Present lasts continuously, it is something ever new which endures and

In what follows we are interested in the non-Cantorian approach of the Alternative Set Theory (Vopěnka, Sochor, Pudlak and others of the Prague School) and that of the Internal Set Theory (E. Nelson) to countable and uncountable infinity. This involves the reduction of classical continuity and openness to a shift of the "horizon" of countability in Alternative Set theory (AST) or the presence of an external to Cantorian Set theory predicate *standard* in Internal Set theory (IST). It should be reminded that in Cantorian theory uncountable infinity is introduced by an axiom proved to be independent of the other axioms of ZF system in the well-known Continuum Hypothesis.

By an appropriate outline of the respective approaches in sections 2 and 3 we'll try to make clear that their view of continuum stands essentially in the reduction of the idealized (ε,δ) continuity of real numbers and the relevant notions of analysis and topology to a hereditary finiteness of natural "appearances" to the horizon of "observability" and the assumption of certain extension principles in AST or "unknown" predicates and relevant axioms beyond the fixedness of the ZFC system[2] in IST.

We further hold that these approaches incorporate in mathematical axiomatization the impredicativity of the absolute subjectivity of the flux of conscience in the constitution of spatiotemporal phenomena. This ultimate irreducibility is put into evidence in radical reduction of the flux in the Husserlian sense (see Patočka, 1992, VII, pp. 165–168). We should keep in mind Husserl's fundamental thesis of the genetic-kinetic constitution as the mode in which objects appear within the temporal flow of our experience, the temporal approach being crucial to our understanding of human beings and cultural objects (Moran 2000, Ch.5, p.166). In this way the naturally intuited continuity of physical processes is essentially reduced to the unity of the flow of their multiplicities in the constituting flux of conscience.[3]

Both these theories adopt an "observer" approach to the constituted continuum of our experience, the AST theory in a more closely phenomenological fashion by adopting the "shift of the horizon" (or prolongation) axiom to reach the vagueness of infinity beyond the naturally intuited hereditary finiteness of the horizon of countability. Horizon which, according to P. Vopěnka, limits our capacity for

changes in consciousness." It should be further stressed that Weyl's view of intuitive continuum in Das Kontinuum (Ch. 2, sec. 6), back in 1918, was largely based on the Husserlian descriptions of the consciousness of internal time. This was also true for L.E.J. Brouwer to the extent that his early ideas about the primordial intuition of Mathematics are readily understood in connection with Husserl's phenomenological description of internal time, see (Van Allen *et al.*, 2002, pp. 203–205).

[2] *Zermelo-Fraenkel system of Cantorian Set theory plus the Axiom of Choice.*

[3] In the comments of Dorian Tiffeneau to Husserl's *La phénoménologie et les fondements des sciences*, "the real being is not given but as a unity of multiplicities; the kinetic method studies how these unities are constituted progressively by restituting them in the flux of multiplicities" (Husserl 1993, Notes du Chapitre I, p. 211 transl. of the author).

observation and distinction in all directions, of course not only in the optical sense, but in the Husserlian sense as understood in E. Husserl's "*Krisis der europäischen Wissenschaften und die transzendentale Phänomenologie*" (Husserl, 1970).

In 1.1 we note a circularity of the notion of continuity as was used by Husserl himself in the description of the flux of appearances in conscience whereas in 1.2 we see that the double intentionality of the flux of conscience can be interpreted by the gluing operators in the mathematical modelization of Jean Petitot (Petitot, 1999).

Finally, in conclusion, we evaluate the capacity of these alternative theories to provide a fertile interpretational frame linked to a mathematical-phenomenological "naturalization".

1. THE PHENOMENOLOGY OF THE CONTINUUM

1.1. IRREDUCIBILITY OF THE RADICAL PHENOMENOLOGICAL REDUCTION

It is worth noting that the phenomenological analysis is of a kinetic (*kinetisch)* and not an institutional *(katastematisch)* character[4]. Moreover the conviction to an objective reality in an absolute sense is substituted – the corresponding Husserlian term is *epoché* – by a constituted reality approach which is of a fundamental character in phenomenological interpretation. The constituted objects are *immanent* to the constituting flux of conscience in which they are reflected in a certain mode, that of the *vor-zugleich* (anterior-simultaneous) which entails a continuum of phases trailing behind an original sensation and each of which is a retentional conscience of the preceding "present" (Husserl, 1996).

This temporal conscience of immanences is the unity of a whole, an all encompassing unity of the simultaneity and anteriority of the original sensations of actuality that transforms continuously every group of original sensations in the simultaneity to a trailing into an immediate posteriority which is a continuity and each of whose points is in the form of a homogeneous flow.[5]

[4] See, E. Husserl «*La phénoménologie et les fondements des sciences*», Ed. PUF, Paris 1993, App. I, §6, p. 158, transl. of the author: "*The mode of ontological consideration is so to say catastematic [Katastematisch]. It takes the unities in their identity and in regard to their identity, as something fixed. The phenomenological and constitutive consideration takes the unity in the flux, which means as a unity of a constituting flux, it is attached on the movements, on the flows in which such a unity and every component, aspect or real property of this unity is correlate of the identity*."

[5] "The totality of the group of original sensations is bound to this law: It transforms into a constant continuum (in ein stetiges Kontinuum) of modes of conscience, of modes of being-in the flow and in the same constance, an incessantly new group of original sensations taking originally its point of depart, to pass constantly (stetig) in its turn in the being-in the flow. What is a group in the sense of a group of original sensations, remains in the modality of the being-in the flow" (Husserl, 1996, §38 p. 102. transl. of the author).

Let us now bear a critical look to these assertions and see how we can interpret in an ontological or in a phenomenological kinetic fashion the continuous mode of the anterior-simultaneous flow of the original sensations with the "queue" of their retentions in conscience. E. Husserl responds to this problematic by appealing to what he calls double intentionality of the retention in the flux of conscience, namely the immediate retention of the immanent object in the flux of conscience (the sonore effect of a sound, for example) on the one hand and the intentional constitution of the "descending" sequence of retentions of this primary sensation in the flux as a continuous unity always in the anterior-simultaneous mode of flow. That is, each new continuity of phases which present themselves instantaneously in simultaneity is a retention with respect to what is group continuity in simultaneity in the anterior phase. "*Thus, the flux is traversed by a longitudinal intentionality which, in the course of flux, overlapps itself continuously*" (Husserl, 1996, §39 pp. 106–107, transl. of the author).[6]

However, in this retentional-protentional mode of the constitution of the flux in itself lacks a clear definition of the term continuity as is described modally in a somehow circular sense in terms of the constituted unity of the flux; further, this self-appearance of the flux as a phenomenon in itself is not but an objectivation of what is the ultimate subjectivity, the absolute Ego, that is, the absolute subjectivity of the flux of conscience. This is the ultimate and most radical phenomenological reduction which can be regarded as the key to comprehend the inherent vagueness of the notion of continuity, even in the kinetic terms of the constituted reality in Husserlian sense. For Husserl himself asserts that it is impossible to extend the phases of this "flux" in a continuous succession, to transform it mentally in a way that each phase "extends" identically on itself, a certain phase of it belonging to a present that constitutes or to a past that also constitutes (not constituted), to the degree that it is an absolute subjectivity beyond any predicate and whose retentional continuity in the constituting flux is not but its objectivation, its ontification by its "mirror" reflexion (Husserl, 1996, §35, p. 98).

It is clear that what is being intuited as a continuous flow in the temporal constitution of a group of simultaneities and corresponding retentions is irreducible in terms of an ontological deconstruction to its constituent parts essentially being the objectivation of an inherently elusive process which is always "on going" and every effort to represent it predicatively or even simply reflect on it produces its "mirror" objectivation. This phenomenological ultimate irreducibility in the flux of time consciousness reached essentially by the inherent impredicativity of constituted continuity we'll try to put into evidence in the axiomatization of

[6] "*If we consider an arbitrary phase in the flux of conscience (in which phase appears a sonore present and a fragment of the sound duration in the mode of just-passed by) we see that it comprises a continuity of retentions that possesses a unity in the Vor-zugleich*." (E. Husserl, 1996, §39, p. 106, transl. of the author).

Alternative and Internal Set theories as non-Cantorian versions of nonstandard theories in sections 2 and 3.

From the development of these sections it will also be made clear how these theories offer a more natural mathematical approach to processes in life as at least P. Vopěnka claims that AST does. In general terms, AST tries to do it by following the unfolding of hereditarily finite appearances of phenomena to the horizon of "observability" and describing axiomatically vagueness beyond it. But let us keep also in mind what H. Weyl stated in *Das Kontinuum (1918)*, namely that it is an "act of violence" to assume the perfect coincidence of the analytical construction of the continuum with that of phenomenal space and time.[7]

It is in the line of this aphorism that non-Cantorian theories as well as intuitionistic ones follow an alternative approach to the notion of continuum.

1.2. VAGUENESS IN PHENOMENOLOGICAL KINESTHESIA

Before dealing with the question of continuity in Jean Petitot's modelization of the kinesthetic control of perception in the constituted reality, we refer to Husserl himself as to what he regards as parallel problems, that is the constitution of the universal and unique space which is co-perceived in each specific perception to the extent that everything perceived appears as residing in it corporeally and the constitution of the unique time in which is inserted the temporality of all things their duration and the duration of their processes as well as every corporeal ego and also its "psychic experiences" (Husserl, 1996, suppl. X, p. 161). Further on, the problem of kinesthetic control of perception belongs to "*the great task ...of penetrating as deeply as possible into three-dimensional phenomenological "creation", or, in other words, into the phenomenological constitution of the identity of the body of a thing through the multiplicity of its appearances*" (Husserl, 1973, §44, p. 154, transl. of the author).

The kinesthetic control of perception is not only a presupposition for the effective identity of the appearing object founding thus logical identity upon continuous variability and thus synthetic a priori laws to continuous synthesis which is, in fact, kinetic synthesis. It also "*rules phenomenologically, temporal series corresponding to three classes of movements, namely, those of the eyes, the body and objects*" (Petitot, 1999, 6.3.1. p. 354). It is essential too in interpreting phenomenologically the source of each movement as something "internal" to these Husserlian kinesthetic sensations. As we'll proceed now with the kinesthesiological analysis of the simplest situation, that of a body (subject) being fixed and objects remaining at rest, it will become evident how vagueness is "interwoven" to the

[7] "*… the continuity given to us immediately by intuition (in the flow of time and motion) has yet to be grasped mathematically*" (Weyl, 1977).

unity of the constituted movement based on the temporal discreteness of the correlations $k_{1\leftrightarrow}i_1$, ..., $k_n\leftrightarrow i_n$ . .This particular situation reduces to the purely ocular kinesthetic sensation schematized by the correspondence k↔i between the space of kinesthetic controls K and the space of visual images F by applying a temporal parametrization through reciprocally corresponding paths $k_{t,}$ i_t.

Jean Petitot refers to the elementary example used by Husserl to describe more precisely the nature of the link between k_t and i_t (the temporal paths of kinesthetic sensations and those of image variations) and also that of the "fixed association" of K to the visual field M which is modelized as a simple domain D (a two-dimensional disk). For a more detailed description, see Petitot, 1999.

To each point p=a, b, c, d of the square S in Fig. 1 corresponds a token D_p of the field D as a way of "interpreting" the focusing on each such point.

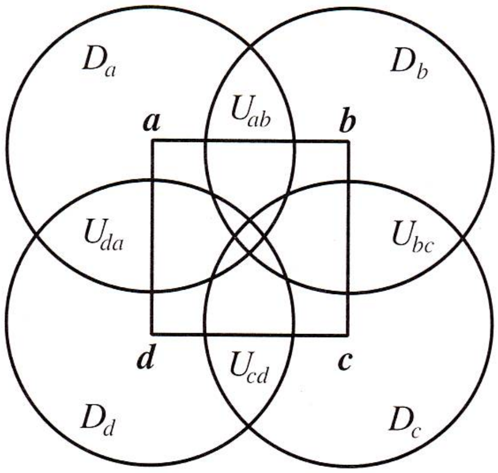

Fig. 1

Quoting from Jean Petitot: "*if the figure i_α filling in D_α can "refer" to the figure i_β filling in D_β, it is because D_α and D_β overlap, and are glued together through their intersection $U_{\alpha\beta}$=$D_\alpha \cup D_\beta$. This means that there exists a local gluing isomorphism $\varphi_{\alpha\beta}$: $U_{\alpha\beta} \subset D_\alpha \rightarrow U_{\alpha\beta} \subset D_\beta$ identifying the intersection $U_{\alpha\beta}$ viewed as a subdomain of D_α with the same $U_{\alpha\beta}$ viewed as a subdomain of D_β. In the continuous limit, there exists a temporal series D_t with gluing operators φ_{tt}' for t and t′ sufficiently near. This spatiotemporal series is filled in by the image series $i_{t.}$ To say that the "pointing" of each i_t to other $i_{t'}$ is intentional, or that intentions "go through" the series i_t, is to say that intentionality corresponds to gluing operators identifying different points of the visual flow as the same. .... More precisely, intentionality corresponds to the realization in consciousness of the gluing operators. Once again, it is essential here not to confuse, as the natural attitude does, the constituting level and the constituted one…. This is the main role of kinesthetic controls: the k_t are gluing protocols.*" (Petitot, 1999, 6.3.7., pp. 356–357).

This modelization that reduces the kinesthetic constitution of movement, purely ocular in our instance, to gluing operators k_t realized in conscience for t,t′ sufficiently near, is a "transformation" in a mathematically meaningful fashion of the idea of longitudinal intentionality of the flux of conscience. As was the case in the retentional mode of the constituting flux, here it becomes clear too that in the phenomenology of movement through kinesthetic controls one cannot avoid the circular introduction of the notion of continuity in the constituted unity of the multiplicity of appearances. In this case, this continuity factor is represented by the local gluing isomorphism $\varphi_{tt'}$ for t and t′ sufficiently near which glues together the temporal series D_t filled in by the image series i_t in the continuous whole of constituted reality. We can infer in the phenomenological perspective of the constituted spatiotemporality a sequence of immanences of original sensations in the flux of conscience together with a vagueness of "indiscernibilities" or infinitesimalities in-between, constituted as a unity by longitudinal intentionality in the *vor-zugleich* mode of the flux or by the gluing protocols k_t in the referential example of Jean Petitot.

Before dealing with indiscernibility or vagueness from the standpoint of Alternative and Internal Set theories in sections 2 and 3, we find it purposeful to refer to the Husserlian idea of scale invariance, as evident generic similarity, which can lead to minima visibilia as point-like ultimate minimalities bearing the same eidetic relationships "discovered" in the macroscopic universe, (see Husserl, 1973, §48, p. 166).This idea seems to have a profound effect on the shift of the horizon principle embodied in AST theory.[8]

2. THE PHENOMENOLOGICAL RELEVANCE OF THE AST APPROACH

We referred in the previous section to the inherent vagueness or indiscernibility characterizing the constituted in terms of the flux of conscience phenomenological continuum. We came to describe a process "interweaving" a discreteness of immanent multiplicities in conscience with a vagueness of "indiscernibilities" filling in-between. At this point it seems purposeful to quote from C. Zeeman's "*The topology of the brain and visual perception*", that "*nothing in physics suggests the existence of so sophisticated a mathematical construction as the real numbers system…. Nothing in physics suggests even non-countability. Surely we would have a more natural approach to the foundation of physics by postulating only the existence of discrete fundamental parts (which are otherwise*

[8] In the sense that this evident generic similarity justifies the transposition of the eidetic relationships "discovered" in the universe of common intuition to that beyond this "horizon". It is remarkable, though, that P. Vopěnka seems to deny this principle in Vopěnka, 1991, p. 123, where he insists that all ideas held hitherto could collapse beyond some genuinely qualitative shift of the horizon.

undefined) without embedding them in any so-called ether. We should then postulate certain combinatorial laws describing which particles are permitted to be within tolerance." (Zeeman, 1962)Tolerance is defined here as an indiscernibility-like, binary relation on a set X that is reflexive and symmetric.[9]

Now we'll draw our attention on how AST reduces classical sense continuum of topological shapes and motions to the extension by the use of the axiom of prolongation of classes of finite segments of natural numbers to class infinities transcending the "horizon of observation". In this sense a topology can be defined relying basically on the notion of countable classes in the extended universe of sets and the axiom of prolongation not adopting thus the traditional approach of topological openness, connectedness etc. based on the continuity of the real number system.

AST, as exposed in its fundamentals by P. Vopěnka (Vopěnka, 1979), assumes a universe of sets formed by sets constructed iteratively from the empty set together with some axioms subjecting the sets of this Universe to laws valid in Cantorian set theory for finite sets excluding "abnormal" circularities like the Set of all sets. This universe is extended by the inclusion of classes of the form {x; φ(x)} where φ(x) is a property of sets from the universe of sets. Classes that are not sets are called proper such as the universal class ∏.

We proceed now to the idea of countability in the sense of hereditary finiteness that is closely related to the idea of observation towards the "horizon".[10] We need only know in advance that, roughly, a segment of a class is a subclass with respect to linear ordering that contains with each of its elements all its predecessors.

Formally one has two definitions:

– *A pair (A, ≤) of classes is called an ordering of type ω iff:*
  (1) ≤ linearly orders A
  (2) A is infinite and
  (3) for each $x \in A$, the segment $\{ y\in A; y\leq x \}$ is finite.
– *A class X is called countable iff there is a relation R such that (X, R) is an ordering of type ω. A class is uncountable iff it is neither countable nor finite.*

In case one goes beyond the horizon of countability, the "horizon of observability" in the phenomenological attitude of Husserl's *Krisis*, one has to adopt the following Prolongation (or shift of the horizon) axiom:

[9] An interesting point of view regarding the problematic of elementary particles individuality in Quantum Mechanics is offered in Lavine, 1991.

[10] By the words of P. Vopěnka "if a large set x is observed then the class of all elements of x that lie before the horizon need not be infinite but may converge toward the horizon. The phenomenon of infinity associated with the observation of such a class is called countability" (Vopěnka, 1979, Ch. I, p. 39).

For each countable function F there is a set function f such that $F \subseteq f$.

We should notice that the Husserlian idea of scale invariance is a conceptual presupposition for this axiom. It is also important to underline that P. Vopěnka makes a fundamental distinction between the class of all finite natural numbers FN proved to be a countable class and the set-theoretically definable proper class N of all natural numbers which is uncountable. In a witnessed universe, that one that adopts the viewpoint of an observer incorporated in it (thus essentially a phenomenologically intersubjective universe) "*The classical natural numbers correspond to elements of N, whereas FN forms a canonical representative of the way to the horizon.*" (Vopěnka, 1979, Ch. II, p. 63)

Then one can define a topology by the Kuratowski closure operations which are not taken as primitive as is the case in classical topology but defined instead in terms of indiscernibility equivalences $\doteq$ that underlie every topological definition and are fundamentally based on the shift of countability (prolongation) axiom in the AST sense. The underlying idea in the definition of an indiscernibility equivalence is that in each infinite set of "observed" objects there must be at least one pair (x,y) of mutually indiscernible elements, in formalism $x \doteq y$.

By defining a class X to be a figure iff X contains with each of its elements x all y such that $x \doteq y$, two classes X, Y to be separable – (Sep (X, Y)) – iff there is a set-theoretically definable class Z such that Fig $(X) \subseteq Z$ and Fig $(Y) \cap Z = \emptyset$, and the closure of a class A, $\bar{A}$, to be $\bar{A} = \{x; \neg \text{Sep}(\{x\},A)\}$, P. Vopěnka defines a topology in the extended AST Universe with the notion of indiscernibility as conceptual and formal foundation for all subsequent topological constructions including the definition of the idea of motion (Vopěnka, 1979, Ch. III, IV, pp. 87–88 and 98–108).

It seems worthwhile mentioning the approach of A. Sochor and A. Vencovská in "Indiscernibles in the Alternative Set theory" defining a class of indiscernibles as a class of natural numbers such that there are no two finite increasing sequences of its elements that can be distinguished using a set formula without parameters and proving that there exists a class of indiscernibles which is a Π-class and which is not a semiset.[11]

A class is defined as real iff there is an indiscernibility equivalence such that the class in question is a figure in this equivalence. Further, it is proved that if X is a real class, then there is either a set U with $U \subseteq X$ and $U \mathrel{\hat{\approx}} \alpha$ (numerically equivalent to $\alpha \in N$) or for each $\gamma \in N\text{-FN}$ there is a set $U \supseteq X$ with $U \mathrel{\hat{\approx}} \alpha\gamma$ (Sochor *et al.*, 1981, pp. 789–790).

[11] The notion of proper semisets is fundamental to AST theory. They are proper classes inside very large sets whose existence is guaranteed by an axiom "external" to the AST extended universe of sets. They bear grosso modo the traits of "fuzziness" or "non surveyability", see Vopěnka, 1979, Ch. I, Section 3, pp. 33–36.

Evidently the classical real continuity is reduced to the uncountability of the class of natural numbers N in the AST sense.

We can conclude that indiscernibility relations in AST sense modelize vagueness or "blurring of vision" in topological structures occurring as we transcend the horizon of countability of finite natural numbers to the uncountability of the infinite class of natural numbers. Moreover, we can assert that indiscernibility equivalences in AST act in principle as the gluing operators in Petitot, 1999, gluing points of classes beyond the horizon of AST countability or observability in Vopěnka's phenomenological attitude.

3. THE INTERNAL SET THEORY APPROACH TO CONTINUITY-VAGUENESS

It is our intention now to sort out the main conceptual and axiomatical characteristics of the Internal Set theory's approach to the key ideas of continuity and vagueness in view of its adoption of an external to ZFC Set theory and undefined unary predicate "standard" involving indirectly the presence of an observer in the classical Cantorian universe. "*The intensional development of a large part of nonstandard analysis essentially coincides with Nelson's Internal Set theory, appropriately interpreted*" and "*In the intensional development of nonstandard analysis, infinitesimals and infinitely large numbers do not exist in an objective way as in the extensional case, but rather their existence has a subjective meaning and is related to the observational limitations of an "observer"*" (Drossos, 1989). In fact, the introduction of the undefined predicate "standard" in E. Nelson's theory underlies the formal apprehension of a factor of vagueness connected to a series of "observations" carried out in a discrete mode. It is suggested that: "finiteness" + "vagueness" = "unlimited", where "unlimited" is a non-Cantorian equivalent to infinity.

In general, a vague predicate R is obtained in case we have a series of "observations" $O_0, O_1, \ldots, O_n$ such that:

(iii) $R(O_0)$

(iv) $R(O_i) \Leftrightarrow R(O_{i+1})$, i=0,1,…, n–1 that is O_i and O_{i+1} are indistinguishable with respect to R, and,

(v) $\neg R(O_n)$

Using the Transfer principle of IST and appropriate theorems within IST framework one can prove that the predicate "standard"– abbreviated as st – is a vague predicate in the set N of natural numbers:

(i) st(0)

(ii) $st(n) \Leftrightarrow st(n+1)$

(iii) There exists a nonstandard $n \in N$ (Drossos, 1989, theor. 4.1., p. 295).

In the proof of this theorem an essential use is made of a result that is produced straightforward using the IST Idealization principle, namely that every infinite set contains a nonstandard element. In particular, there exists a nonstandard natural number (Nelson, 1986, 1.3. p. 5).

It is of prime importance to stress the intuition behind the Idealization principle which alongside the Transfer and Standardization principles (I, S and T principles) are the axiomatical pillars of Internal Set theory: "*The intuition behind Idealization principle is that we can only fix a finite number of objects at a time. To say that there is a y such that for all fixed x we have A, is the same as saying that for any fixed finite set of x's there is a y such that A holds for all of them*":

$\forall^{\mathbf{stfin}} \mathbf{x}' \ \exists \mathbf{y} \ \forall \mathbf{x} \in \mathbf{x}' \ \mathbf{A} \leftrightarrow \exists \mathbf{y} \ \forall^{\mathbf{st}} \mathbf{x} \ \mathbf{A}$, where A is an internal formula, that is one that does not involve the "unknown" predicate standard even indirectly (Nelson, 1986, 1.3. p. 5).

We can deduce vagueness in the infinity in terms of the predicate "standard" in the set N of natural numbers by relying on the Idealization principle. Thus we can point a common conceptual underlying basis between this principle which together with the Transfer principle induce nonstandard elements *in fine*[12] and the prolongation axiom of AST as an axiomatic means to shift the horizon of phenomenological observability to the vagueness of continuum. It is to be noted here that although E. Nelson insists that predicate "standard" plays a syntactical rather than a semantic role in the theory, it is implicitly taking this latter role by the adoption of the three appropriate axioms I, S and T. The convergence in the conceptual approach is all the more evident since as in AST one defines topological notions with indiscernibility equivalences taken as primitive, so infinitesimality or unlimitedness and hence continuity and openness in IST are defined by taking as primitive the predicate "standard" together with the I,S and T axioms. Without linking, in effect, continuity and topological openness necessarily to the standard real number system as standardness and nonstandardness are not described solely within the real model. We should underline that the predicate "standard" may be associated intuitively with the notion of fixed (concretely grasped) in informal mathematical discourse any object that can be uniquely described within internal mathematics assumed to be standard as, for example, the real numbers R, 0, π, the first uncountable ordinal, etc. (Nelson, 1986, 1.2. p. 4).

The novelty in IST approach in what concerns classical continuity and openness stands in that it treats those fundamental ideas of mathematical analysis and topology by enriching the existing ZFC system with the undefined and related to the implicit presence of an observer external predicate "standard" alongside the appropriate axiomatical equipment with no relevance, in principle, to any particular mathematical model.

[12] The Transfer principle essentially states that if something is true for a fixed but arbitrary x, then it is true for all x, $\forall^{\mathbf{st}} \mathbf{t_1} \ldots \forall^{\mathbf{st}} \mathbf{t_n} \ [\forall^{\mathbf{st}} \mathbf{x} \ \mathbf{A} \leftrightarrow \forall \ \mathbf{x} \ \mathbf{A}]$, where A is again an internal formula.

We can define common ground between AST and IST consisting in the shift of the bounds of hereditarily finite countability (AST) or fixedness (IST) to the vagueness of infinity by the adoption of appropriate (or rather *ad hoc*) axioms or predicates external to a first-order language axiomatical system. We support that it is the impredicative character (in an ontological sense) of the most radical reduction in the phenomenology of the constituting flux of conscience that is reflected in the mathematical axiomatization of continuum in these non-Cantorian theories.

This is essentially the case in intuitionistic approach, too. In both Brower's and Weyl's approach, intuitive continuum is grasped by describing the shift to an indefinite horizon in terms of *ad hoc* extension axioms beyond the natural bounds of the finite and discrete which in the case of choice sequences is represented by their initial segments (Van Atten *et al*., 2002, pp. 220–224). We should last take into account that this impredicativity manifests itself in classical Cantorian system in the adoption of the, independent to the other axioms of ZFC, Continuum Hypothesis (CH) the validity of which is still a topic of debate among set theorists.

Referring further to the introduction of nonstandard elements in superstructures that represent the extensional (or Cantorian) aspect of nonstandard analysis we must, without intending to enter into a deeper and more detailed analysis, underline the fundamental role of Zorn's lemma and consequently its logically equivalent Axiom of Choice, in its stronger form as Global Choice,[13] in ultrapower construction (existence of free ultrafilters Φ_M and their use both in Los´ theorem and the Mostowski collapsing function). To a development of an intuitive meaning of the (uncountable) Axiom of Choice which in the case of nonstandard superstructures reflects the shift from countability to uncountable infinity, we refer to the ideas of Alain Connes in (Connes, 2000, Ch. 1, pp. 17–23).

CONCLUSION

We sought the conceptual link between the Husserlian idea of the restitution of the multiplicities of appearances in the self-constituting unity of the flux of conscience and the mathematical axiomatization of the shift from the discretely intuited as unities in succession to the vagueness and indiscernibility of the continuum. This stands essentially either in the adoption of an impredicative subjectivity in the constituting flux in Phenomenology or the adoption of *ad hoc* "external" axioms or predicates in nonstandard and non-Cantorian theories.

Both AST and IST enrich but do not replace the existing ZFC system.[14] Nonstandard analysis-quoting E. Nelson- supplements but does not replace internal

[13] The class { < x, y>: □(x, y) } is a linear order of all sets which has small initial segments and is a well ordering. □ is a new binary predicate symbol added to the language of ZFC so that this axiom of Global choice holds (Ballard, 1994, Ch. 9, p. 68).

[14] For more constructively or intuitively oriented versions, see respectively, Palmgren, 1995 and Lano, 1993.

mathematics. In fact, it is proved that Internal Set theory is a conservative extension of ZFC system in the sense that every internal theorem of IST is a theorem of ZFC and that Alternative Set theory is a conservative extension of ZF_{Fin}.[15] It should be noted here Gödel's view of the concepts and axioms of classical set theory in his well-known article *What is Cantor's Continuum problem*, namely, that the undecidability of Cantor's conjecture (the Continuum Hypothesis, which was proved to be so by K. Gödel and P. Cohen) "can only mean that the axioms known today (the axioms of ZFC) do not contain a complete description of reality" (Gödel 1947, p. 520).

Each tries to express in mathematical formalization the vagueness that becomes evident in the constitution of the external to the conscience of the subject reality by adopting a more phenomenologically oriented attitude in the description of the horizon to the continuum and its underlying indiscernibility. Equipped – and in that they resemble less to Husserl's view of classical mathematics as an exact science of pure idealities in *Ideen I* – with a measure of vagueness and ultimate irreducibility inherent in the intuition of those fundamental concepts in the field in which they become meaningful, the field of our intersubjective *lebenswelt* in its ever shifting horizon. Vagueness that reflects our inability to describe continuity in an ontological sense and handle it mathematically in the same first-order language (without adding extra *ad hoc* axioms or undefined predicates) as that of the natural numbers hereditarily finite countability in our witnessed Universe. As Giuseppe Longo put it (Longo, 2001): "*...as for geometry, and following Riemann, Poincaré, Weyl, we referred to symmetries, isotropy, continuity and connectivity of space, regularities of action and movement, as 'meaningful properties'. They are meaningful as they are embedded in our main intentional experience as hinted above: life.*"

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[15] ZF_{Fin}, stands for the Zermelo-Fraenkel system with the exception that the axiom of infinity is substituted by its negation. For further details, see Nelson, 1977, theor. 8.8. and Sochor, 1983, §9, p. 145.

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