COHOMOLOGY, PRECOHOMOLOGY, LIMITS AND SELF-SIMILARITY OF CONSCIOUS ENTITY

(Sheaf Theoretic and Categorical Formulation of Consciousness)

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PART A. BASIC IDEAS AND REVIEW

In our earlier work, [K-S. Sheaf] and [K-S. Tokyo], we introduced the notion of *conscious universe* $\underline{\boldsymbol{U}}$ (or the *sea of consciousness*) as a category of presheaves. More precisely, $\underline{\boldsymbol{U}}$ is the category of contravariant functors from the category $\boldsymbol{T}$ associated with a topological space **T** to a product category $\prod_{\alpha\in\Gamma} C_\alpha$ of categories, where Γ is an index set. The category $\boldsymbol{T}$ is said to be the *generalized time space* (or *generalized time category*). A contravariant functor P in $\underline{\boldsymbol{U}}$ is said to be a presheaf defined on $\boldsymbol{T}$ with value in $\prod_{\alpha\in\Gamma} C_\alpha$. Namely,

$$\text{(A.1)} \qquad \underline{U} = \left(\prod_{\alpha\in\Gamma} C_\alpha\right)^{T^{opp}}$$

To be more explicit, for an object V in $\boldsymbol{T}$, i.e, an open set V of **T**, and for an object P in $\underline{\boldsymbol{U}}$, we have $P(V)=\left(P_\alpha(V)\right)$, $\alpha \in \Gamma$, where each $P_\alpha(V)$ is an object of C_α. Recall also that a *conscious entity* is a presheaf P in $\underline{\boldsymbol{U}}$, where $\{C_\alpha, \alpha \in \Gamma\}$ represents the totality of mental and physical categories of a conscious entity. Note also that some of the categories in the product are discrete categories with structure, i.e, categories with no morphisms but with specifically given structures in those categories (see Part C). We assume that the real line **R**, corresponding to *time*, is embeddable in **T**. In the program which will be described in Part C, it may be important to consider **R** is associated to each P. Namely, it should be written as R_P . Then there is an isomorphism from R_P to R_Q , where P and Q are conscious entities, i.e, objects of $\underline{\boldsymbol{U}}$. Let i be an embedding from **R** to **T**. Then i induces a functor from the category of presheaves on $\boldsymbol{T}$ to the category of presheaves on **R** denoted as i^{-1}. (See [G-M] for operations among sheaves.) That is, for P in $\underline{\boldsymbol{U}}$,

$i^{-1}(P)$ is a presheaf on **R**, i.e., the restriction of *P* on R. One often writes $i^{-1}(P)$ as $P|_R$. There are two different kinds of consciousness (or unconsciousness) in the usual sense. The first kind is *awareness,* which is simply, in this sheaf theoretic definion of consciousness, $P(V) = \left(P_\alpha(V)\right)$, $\alpha \in \Gamma$, in the category $\prod_{\alpha\in\Gamma} C_\alpha$. In Zen philosophy, one begins with the concept of being here and now. Then one reaches the stage of having no thoughts, i.e., each component of $P(V \cap R)$ in each category C_α is a trivial object. We will return to this topic in Part C. (As an elemental introduction to Zen, one may read [S.S.].) The second kind of consciousness is *attention*. When one has a thought on a certain topic, it is the component $P_\alpha(V)$, the image of the projection from *P(V)* in $\prod_{\alpha\in\Gamma} C_\alpha$ to a particular category C_α where the thought occurs.

Now we should answer the following natural questions for this sheaf and category formulation of consciousness.

"Why Category?"

One's cognitive awareness has clear existence, as Rene Descartes said, "I think. Therefore, I am." However, for a conscious entity *P*, a certain component $P_\alpha(V)$ of the awareness *P(V)*, for a generalized time period *V*, need not consist of elements. That is, it is just an object in the category C_α without elements. Hence, the general notion of an object of a category is needed. When there are elements in an object, they are said to be *thoughts*. See [K-S. Tokyo].

For two objects *P* and *Q* in ***U***, namely, two conscious entities, the communication from *P* to *Q* in a category *C* is a correspondence from *P* to *Q*. Note that for the sake of simplicity, we did not index *P* and *Q*, i.e, we regard *P* and *Q* in the category *C* as the *C*-components of *P* and *Q* in ***U***. That is, for *U* and *U'* in the generalized time category ***T***, the information *P(U)* for the generalized time *U* is communicated to *Q(U')* over *U'* by a morphism $P(U) \longrightarrow Q(U')$ in the category *C*. This type of communication is said to be a *horizontal communication* in [K-S. Tokyo]. When *U=U'*, such a morphism from *P(U)* to *Q(U)* is a *natural transformation* in the usual sense from functor *P* to functor *Q*. A *vertical communication* is an information flow from *P* to *P*. Namely, for an object *U* in ***T***, a vertical communication within *P* is an assignment from category C_α to category C_β defined by $I^\alpha_\beta : P_\alpha(U) \longrightarrow P_\beta(U)$. For example, when a conscious entity *P* studies a certain mathematical field C_α to understand another field C_β, this vertical communication $I^\alpha_\beta : P_\alpha(U) \longrightarrow P_\beta(U)$ is an interpretation of the information P has in the category C_α as the informatition in the category C_β. Then, as shown in [K-S, Tokyo], for a horizontal communication of information in C_α, I_β^α induces a horizontal communication in C_β. (See [M-B.], [B-D.], [S.H.] for category theory, where [S.H.] treats categorical sheaf theory as well.)

"Why sheaf?"

Especially in the study of algebraic geometry and complex analytic geometry, sheaf theory and sheaf cohomology theory have been used to connect local properties to global properties. As is described in [K-S. Tokyo], in this formulation, sheaf theoretic restriction map ρ_U^V from *P(V)* to *P(U)* is interpreted as an understanding map in a category. Namely, if a section *s* in *P(U)*, which is called a *thought*, is obtained as *s*= ρ_U^V *(s')*, where *s'* is a section of *P(V)*, then section *s'* is an understanding of section *s*. One can formulate the notion of a unique understanding and also a misunderstanding of a thought in terms of sheaf language. An extension problem, namely, for a given thought *s* in *P(U)* whether there exists a thought *s'* in *P(V)* so that ρ_U^V may map *s'* to *s*, (that is, whether ρ_U^V is epimorphic) or not is an important question. When such an *s'* exists, *s'* is said to be an extension of *s*. It is a simple exercise to rephrase such a notion as a unique extension of *s* in terms of consciousness terminology. When it is impossible to extend beyond *P(V)*, then *s'* is said to be a *terminal thought* of *s*. Thus, brain functions from local information to global information correspond to realization of the local information as the restriction of the global information in the above sheaf theoretic sense. See [A.M.], [J.T.C.] or [G-M.] for sheaf theory, and sheaf cohomology which will be needed in Part B.

PART B. FUNDAMENTAL CONCEPTS

In part A, a horizontal communication is a morphism between two conscious entities *P(U)* and *Q(U')* in a category *C*. In general, let us consider a sequence in *C*:

(B.1) $$---\xrightarrow{\gamma} P(U) \xrightarrow{\delta} Q(U') \xrightarrow{\varphi} R(U'') \xrightarrow{\eta} ---$$

such that this sequence forms a cochain complex. Namely, any consecutive composition of morphisms in (B.1) is trivial. In terms of conscious entities, the composite of any consecutive communication is trivial in *C*. Then the cohomology at *Q(U')*, denoted by $H^*(-- \to Q(U') \to --)$, is defined as the *subquotient*

(B.2) $$Ker\varphi\ / Im\,\delta\ .$$

Let us consider special cases of the above sequence next. In the case where there is only one conscious entity *Q*, i.e., the above sequence becomes

(B.3) $$---\longrightarrow 0 \longrightarrow Q(U) \longrightarrow 0 \longrightarrow ---.$$

then the cohomology at *Q(U)* is *Q(U)* itself. That is, the subobject of *Q(U)* which has no influence on any one is the whole *Q(U)*, and no one influence *Q*. Namely, the subquotient $Ker\,\varphi_U / Im\,\delta_U$, the cohomology at *Q(U)*, is *Q(U)* itself. Next, consider the case where there are only two conscious entities involved. That is, the above sequence becomes

(B.4) $$---\longrightarrow 0 \longrightarrow P(U) \xrightarrow{\delta} Q(U') \longrightarrow 0 \longrightarrow ---.$$

Then the cohomology at *Q(U')* is the quotient $Q(U')/Im\,\delta_U$. That is, the cohomology at *Q(U')* is the quotient object obtained by regarding the influence or information *Q(U')* receives from*P(U)* as trivial part of *Q(U')*. On the other hand, the cohomology at *P(U)* is the subobject $Ker\,\delta_U$. In this case, there is no influence from anyone, and the "core" or "private" conscious part is what *P* does not share with anyone.

As one can observe from these special cases, the cohomology at a conscious entity approximates the core and private consciousness of the entity. When one meditates, (without communication to anyone, i.e., *Ker*-part, and closes eyes and listens to nothing, namely, not influenced by anyone, i.e, *modulo Im*-part), the cohomology represents the real identity of a conscious entity. However, this is merely the first step in Zen meditation. Some of the goals in meditation will be formulated in Part C. In the study of consciousness, it is too strong to assume that sequence (B.1) always forms a cochain complex. Namely, the influence of influence will not be lost in general. One needs a stronger invariant than cohomology for a sequence which need not be a cochain complex. Such an invariant should coincide with the notion of cohomology when the sequence happens to be a cochain complex. From a sequence, which is a not necessarily cochain complex

$$---\xrightarrow{\gamma} P(U) \xrightarrow{\delta} Q(U') \xrightarrow{\varphi} R(U'') \xrightarrow{\eta} ---$$

like (B.1), we construct the following sequence:

(B.5) $$---\xrightarrow{\delta^*} Q(U') \big/ Im\,(\delta \cap \gamma) \xrightarrow{\varphi^*} R(U'') \big/ Im\,(\varphi \cap \delta) \xrightarrow{\eta^*} ---.$$

One can confirm that sequence (B.5) becomes a cochain complex. Then we define the precohomology at *Q(U')* as the cohomology of the cochain complex (B.5), i.e.,

$$Ker\varphi^* \big/ Im\,\delta^* .$$

We write the precohomology as $Ph^*(-- \to Q(U') \to --)$. There is a dual definition for constructing a cochain complex. See [K. Preco] for this construction, the self-duality theorem and related properties of precohomology.

The basic yoga of cohomology (or precohomology) is that the true nature of a conscious entity in a complex of network of communication and influence in a society is the cohomological object, i.e, the subquotient not the object itself. That is, one should consider the *derived category of conscious entities*. See [J.T.C], [G-M] for the theory of derived category.

Next we will consider the notions of inverse limit and direct limit in the context of consciousness. One will notice that the inverse limit of a conscious entity is coherency of conscious entity. Let *P* be an object of $\underline{U} = \left(\prod_{\alpha\in\Gamma} C_\alpha\right)^{T^{opp}}$. That is, *P* is a conscious entity. Then, for *V* in ***T***, *P(V)* is an object of $\prod_{\alpha\in\Gamma} C_\alpha$. Namely, *P(V)* can be expressed as

$$P(V) = (P_\alpha(V))_{\alpha\in\Gamma} \in \prod_{\alpha\in\Gamma} C_\alpha .$$

Conversely, a family of presheaves $P_\alpha : T^{opp} \longrightarrow C_\alpha$, $\alpha \in \Gamma$, determines a presheaf $P : T^{opp} \longrightarrow \prod_{\alpha\in\Gamma} C_\alpha$. That is, we have

$$\underline{U} = \left(\prod_{\alpha\in\Gamma} C_\alpha\right)^{T^{opp}} = \prod_{\alpha\in\Gamma} (C_\alpha^{T^{opp}}).$$

From part A, we have the vertical communication $I^\alpha_\beta : P_\alpha(U) \longrightarrow P_\beta(U)$ within the conscious entity*P*. This communication I^α_β is a typical brain function of the conscious entity *P*. Then I^α_β induces $\overline{I^\alpha_\beta} : C_\alpha^{T^{opp}} \longrightarrow C_\beta^{T^{opp}}$ such that $\overline{I^\alpha_\beta}(P_\alpha) = I^\alpha_\beta \text{n} P_\alpha$. Consequently, we obtain

(B.6) $$---\longrightarrow C_\alpha^{T^{opp}} \xrightarrow{\overline{I^\alpha_\beta}} C_\beta^{T^{opp}} \xrightarrow{\overline{I^\beta_\gamma}} C_\gamma^{T^{opp}} \longrightarrow --- .$$

Then, define an *inverse limit of conscious entities* as

(B.7) $$\varprojlim_\alpha C_\alpha^{T^{opp}} = \{(P_\alpha) \in \prod_{\alpha\in\Gamma} C_\alpha^{T^{opp}} : \overline{I^\alpha_\beta}(P_\alpha) = P_\beta, \alpha, \beta \in \Gamma\}.$$

That is, the inverse limit $\varprojlim_{\alpha} C_{\alpha}^{T^{opp}}$ is a subcategory of the conscious universe $\underline{U} = \left(\prod_{\alpha\in\Gamma} C_{\alpha}\right)^{T^{opp}} = \prod_{\alpha\in\Gamma}(C_{\alpha}^{T^{opp}})$. Let $\pi_{\alpha} : \varprojlim_{\alpha} C_{\alpha}^{T^{opp}} \longrightarrow C_{\alpha}^{T^{opp}}$ be the natural projection to satisfy the universal mapping property. One can also prove $\varprojlim_{\alpha} C_{\alpha}^{T^{opp}} = (\varprojlim_{\alpha} C_{\alpha})^{T^{opp}}$. From the definition (B.7) of the inverse limit, the inverse limit is a collection of vertically well communicated conscious entities. The inverse limit $\varprojlim_{\alpha} C_{\alpha}^{T^{opp}}$ is said to be the collection of coherent or comprehensive conscious entities.

Next, let us consider a direct limit. Intuitively speaking, we make a generalized time period small. For the sequence in a category C as in (B.1)

(B.8) $$---\xrightarrow{\gamma} P(U) \xrightarrow{\delta} Q(U') \xrightarrow{\varphi} R(U'') \xrightarrow{\eta} ---,$$

first take the inverse limit in the category C

(B.9) $$\varprojlim (-- \to P(U) \to Q(U') \to --).$$

Note that the above inverse limit is the usual inverse limit within a category. In terms of consciousness, the limit (B.9) may be said to be the *collective consciousness* (or the *conscious tie*) of conscious entities *P, Q, R,* ---. Next, take the direct limit over generalized periods *U, U', U",* --- simultaneously, then we have

(B.10) $$\varinjlim_{T} (\varprojlim (-- \to P(U) \to Q(U') \to --)),$$

which is called the *germs of collective consciousness* of *P, Q*, ---.

PART C. PROGRAM

Our formulation of consciousness in terms of categorical sheaf theory has flexibility. For conscious entities Q and Q' in $\underline{\boldsymbol{U}}$, we may say that Q is a strictly higher conscious entity than Q' if for V in the generalized time category $\boldsymbol{T}$, whenever $Q(V)$ has the trivial components in categories, then $Q'(V)$ has the trivial components in those categories. It is an interesting question to ask how a conscious machine, if it exists, can be defined in this formulation.

A society or a cultural unity, is a network of complex communications (i.e., morphisms in categories) among conscious entities as shown in Part B. In general, the communication in a society is not a sequence as described in (B.1). In order to obtain a sequence as in (B.1), one can trace the connecting arrows (morphisms) among objects in the category. Then one defines the cohomologies (or precohomologies) for such sequences.

During meditation, it is ideal for one to think nothing. Then cohomological object, i.e., the subquotient, is important as we mentioned. In deeper meditation, it may be said that to make all the components of *P(V)* be final (and initial) objects in categories is even more important. That is, it is the oneness with the wholeness, i.e., final objects in a categories so that there exists a morphism (communication) from (and to) every object in each category. See [K-R] for our sheaf theoretic approach to philosophies. In Zen, one of the fundamental introductory questions is: who is *that I* who asks "who am *I*?". A fractal like equation appears when one formulates this in terms of a sheaf category setting.

In Part A we defined the conscious universe is the category of presheaves, i.e., $\underline{U} = \left(\prod_{\alpha \in \Gamma} C_\alpha \right)^{T^{opp}}$. The totality of conscious entities with thinking or coherent understanding ability in the sense of the definition of a sheaf as described in [K.S. Tokyo] is the subcategory of $\underline{\boldsymbol{U}}$, which may be said to be the ***conscious topos***. It is not clear currently whether one should consider several conscious topoi and their relations, i.e., functors among conscious topoi, associated with many physical universes as described below. The (non-ordered) index set Γ may be divided into three parts. The first part of Γ is used for physical world categories. We will use C_j , *j*=0,1,2,--- $\in \Gamma$ where C_0 is the generalized time category ***T*** itself, C_1 is the micro world, C_2 is the macro world. There exists a canonical embedding from C_1 to C_2. One can ask as a physics problem if there are functors from C_1 to C_2 (or from C_2 to C_1)? For example, non-organic matter M without cognitive functions like non-living things in the usual sense, e.g., a particle, can be considered as a presheaf *M* such that only non-trivial components of *M(U)* are in C_1 and C_2. For example, even if *M(U)* appears at two different locations in any distance apart in C_2, then as long as it is an entity *M*, communication of information between the locations is simultaneous. The second part of Γ is for cognitive categories, including mathematics theories, and the sense of beauty as in various arts. For a conscious entity *P* and a generalized time period *V*, the components in these categories of *P(V)* are the *P'*s awareness or understandingness of e.g., mathematical theory, beauty, etc. Now comes the traditional issue: *Mind and Matter*. We can give a formulation as follows. As an example, let us choose a human for this discussion. A human body is matter consisting of various parts. But each part is matter, however as a whole it poses a mind. Since the matter that involves mind is a brain, we further concentrate on a brain. Let *B* be a brain as an object of ***U*** such that $B = \sum_{\beta \in \Delta} B_\beta$, i.e., a brain consists of various parts B_β. As objects in C_1 and C_2, *B(U)* and $B_\beta(U)$ are non-trivial objects. Then for a generalized time period $V = \bigcup_{i \in I} V_i$ in ***T***, $B(V_i) = (\sum_{\beta \in \Delta} B_\beta)(V_i)$ may be a trivial object in any category other than C_1 and C_2. Here is the place where this *sheaf* theoretic formulation can provide an answer, i.e., as a whole, *B(V)* need **not** be a trivial object in those categories. We consider that the process from $(\sum_{\beta \in \Delta} B_\beta)(V_i)$ to $(\sum_{\beta \in \Delta} B_\beta(V_i))$ is of a neurobiological nature. In various methods for the brain activities, those images are interpreted as the images of functors from the cognitive categories to the physical categories C_1 and C_2. The last part of Γ is a special one, namely let C_ω be the conscious universe ***U*** itself. Let *P* be a conscious entity in ***U***, and let *V* be a generalized time period in ***T*** as before. Then the ω-component of *P(V)* is a conscious entity in ***U***. Namely, $P_\omega(V)$ is an object of ***U***. Hence, it does make sense to evaluate at a generalized time *V'*. That is, one can consider $(P_\omega(V))(V')$, which is an object of $\prod_{\alpha \in \Gamma} C_\alpha$. Then by considering its components in C_0 and/or C_ω repeatedly, one can obtain various ***self-similarity*** equations.

Even though our formulation of consciousness, as a *pretopos*, i.e., the category of presheaves, is a systematic theory of consciousness in the sense that it has a potential to cover physics, mathematics, arts, philosophies and religions, this categorical sheaf formulation needs modification to model true reality. For example, at this moment, we do not know how to define *Yin-Yang* (the male and female) principles in our formulation. Another question is how to formulate the *Big Bang*, i.e., the transition from the trivial category stage of the macrocosm C_2 to the current C_2.

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